Solid Mechanics: Yielding in structural elements

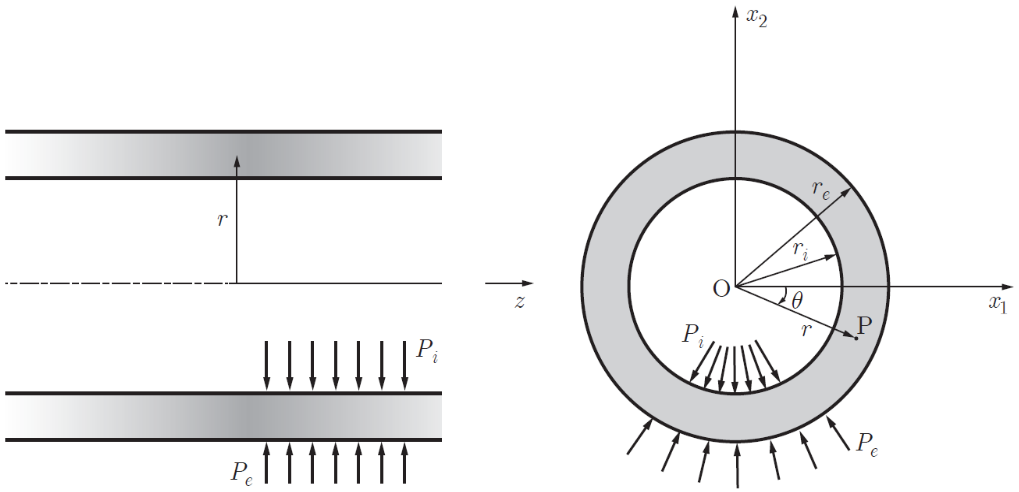
Yielding of axisymmetric structural elements and a simple beam are presented using some well know yield criteria.

From the book: Mechanics of Continuous Media: an Introduction

1. J Botsis and M Deville, PPUR 2018
2. J Botsis, Appendix C: Notes on Plasticity Theory

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure



$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

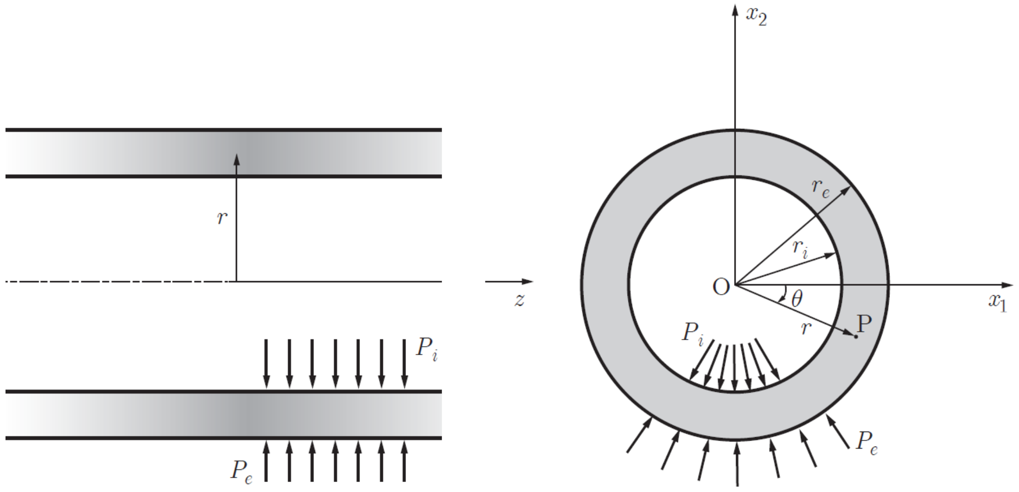
$$u_r = \frac{1-\nu}{E}\frac{r_i^2 P_i - r_e^2 P_e}{r_e^2 - r_i^2}\, r - \frac{1+\nu}{E}\frac{(P_e - P_i)}{r_e^2 - r_i^2}\frac{r_i^2 r_e^2}{r} .$$

Boundary Conditions

$$\sigma_{rr} = -P_i, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_i$$
$$\sigma_{rr} = -P_e, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_e .$$

APPROXIMATION FOR THIN-WALLED CONTAINER

Example: *Hollow Cylinder with Internal and External Pressures*



If the wall thickness is les than 10% of the inner radius, the cylinder is classified as a thin-walled.

The variation of stress with radius is disregarded and the following approximation can be adopted:

$$e = r_e - r_i \quad e \ll r_i$$

$$r_e^2 - r_i^2 = (r_e - r_i)(r_e + r_i) \approx 2er_i$$
$$r_i^2 P_i - r_e^2 P_e \approx r_i^2 (P_i - P_e)$$
$$r_e^2 \approx r_i^2 \qquad\qquad r^2 \approx r_i^2\,.$$

$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$
$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{rr} \approx 0$$
$$\sigma_{\theta\theta} \approx \frac{r_i(P_i - P_e)}{e}\,.$$

Cylinder with internal and/or external pressure

SPECIAL CASES:

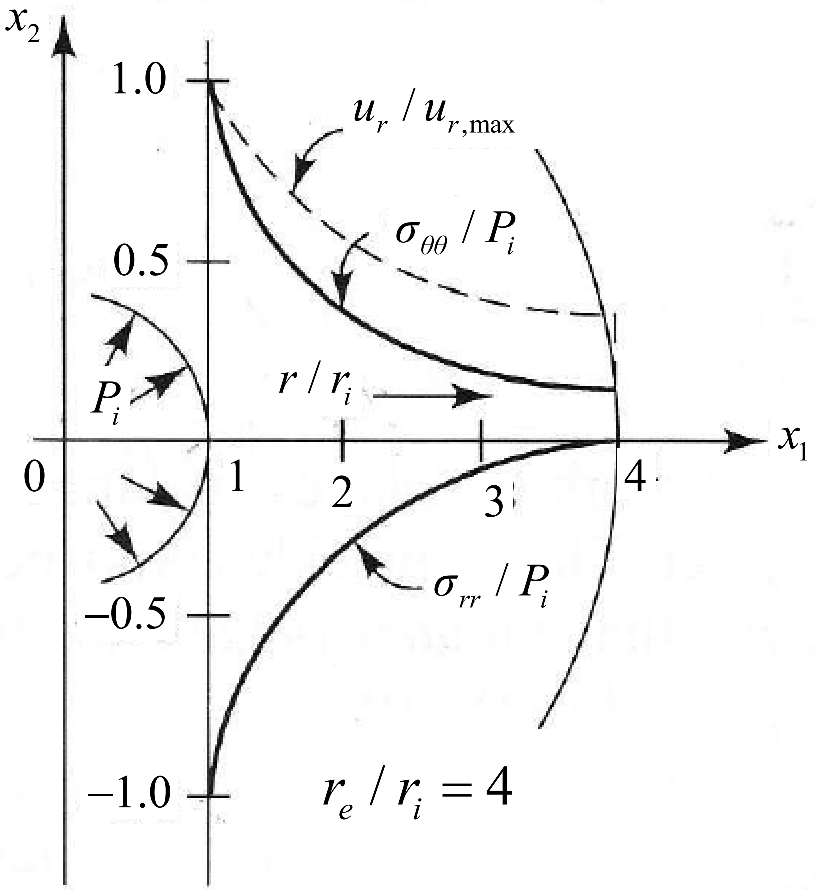
1: Internal Pressure only

The equations we obtained earlier reduce to:

$$\boldsymbol{\sigma}_{rr} = \frac{r_i^2 P_i}{r_e^2 - r_i^2}\left[1 - \frac{r_e^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{r_i^2 P_i}{r_e^2 - r_i^2}\left[1 + \frac{r_e^2}{r^2}\right]$$

$$u_r = \frac{r_i^2 P_i r}{E\left(r_e^2 - r_i^2\right)}\left[\left(1 - v\right) + \left(1 + v\right)\frac{r_e^2}{r^2}\right]$$



Cylinder with internal and/or external pressure

SPECIAL CASES:

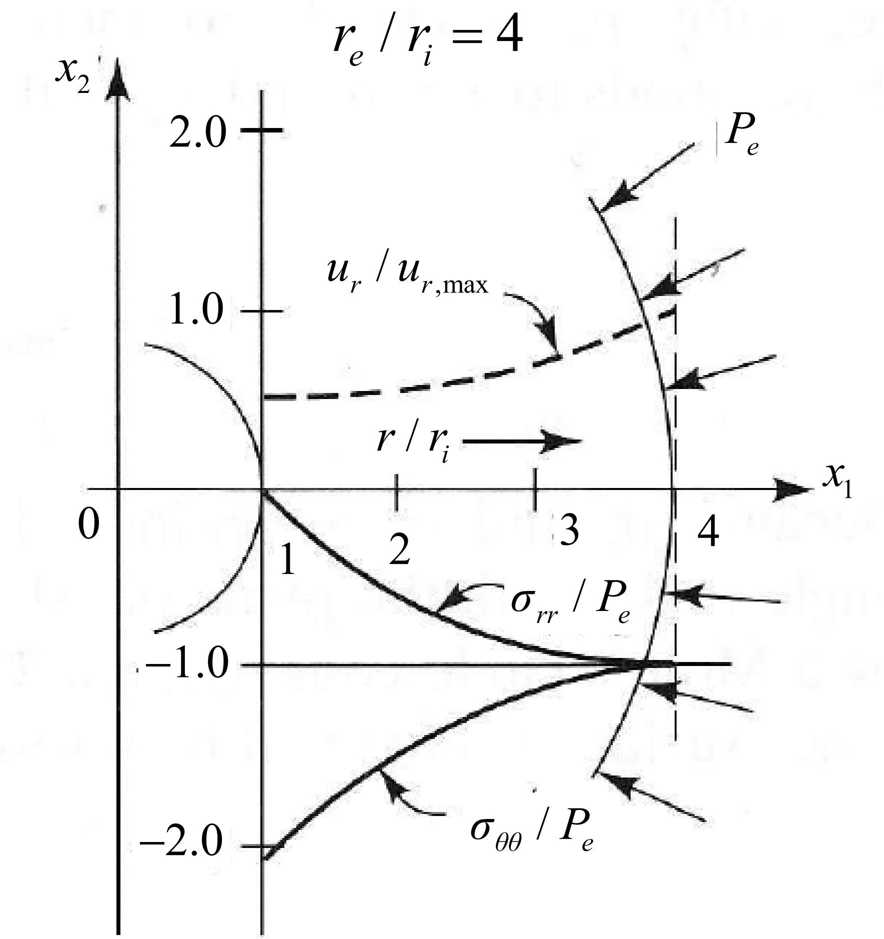
2: External pressure only

The equations we obtained earlier reduce to:

$$\sigma_{rr} = -\frac{r_e^2 P_i}{r_e^2 - r_i^2}\left[1 - \frac{r_i^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = -\frac{r_i^2 P_i}{r_e^2 - r_i^2}\left[1 + \frac{r_i^2}{r^2}\right]$$

$$u_r = -\frac{r_e^2 P_e r}{E\left(r_e^2 - r_i^2\right)}\left[\left(1-v\right) + \left(1+v\right)\frac{r_i^2}{r^2}\right]$$



Mechanics of Solids: Yielding

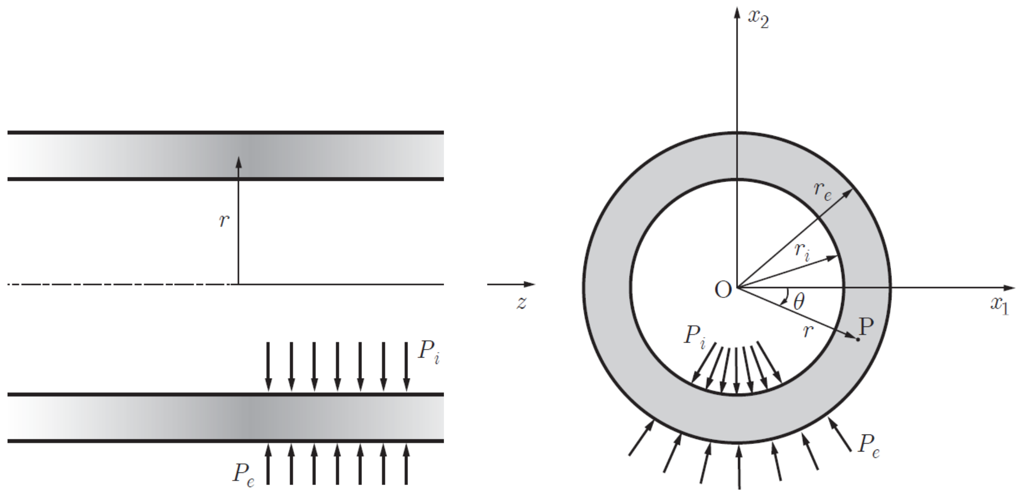
Cylinder with internal and/or external pressure

Elastic Solution

$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$u_r = \frac{1-\nu}{E}\frac{r_i^2 P_i - r_e^2 P_e}{r_e^2 - r_i^2} r - \frac{1+\nu}{E}\frac{(P_e - P_i)}{r_e^2 - r_i^2}\frac{r_i^2 r_e^2}{r}.$$



Maximum shear stress at any point

$$\tau_{\max} = \frac{1}{2}(\sigma_{\theta\theta} - \sigma_{rr}) = \frac{r_i^2 r_e^2}{(r_e^2 - r_i^2)}\frac{1}{r^2}(P_i - P_e)$$

The largest value occurs at the inner surface $r = r_i$

$$\tau_{\max} = \frac{r_e^2}{(r_e^2 - r_i^2)}(P_i - P_e)$$

$$\frac{\sigma_1 - \sigma_3}{2} = \pm\frac{\sigma_Y}{2}$$

At first yield (maxiumum shear criterion)

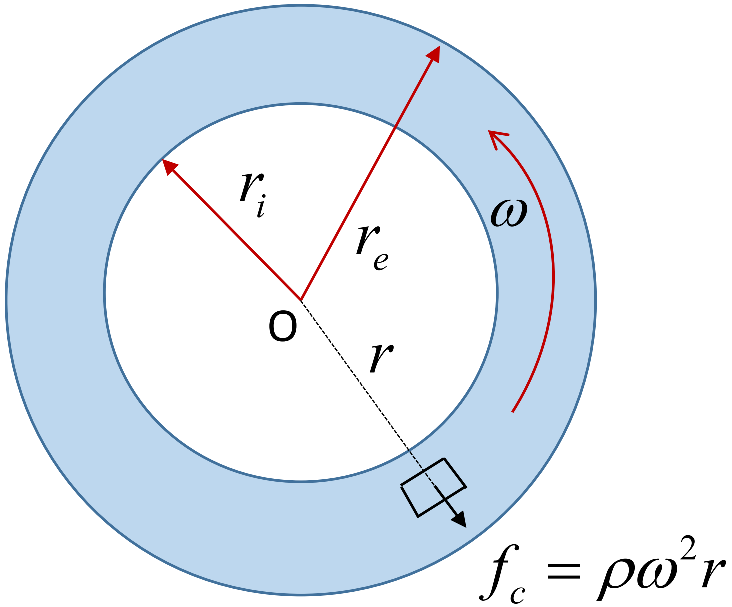
$$\tau_{\max} = \frac{\sigma_Y}{2} = \frac{r_e^2}{(r_e^2 - r_i^2)}(P_i - P_e)$$

(use to dimension the cylinder)

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Disks of constant thickness

(mass density ρ)



ω : angular speed in rad/sec.

We have here a cylindrical symmetry and all stresses are thickness independent.
The equilibrium equation is what we saw earlier with one more term, i.e., the centrifugal force:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0$$

Introduce in it the stresses in terms of displacements,

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right); \quad \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)$$

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = -(1-\nu^2)\frac{\rho\omega^2 r}{E}$$

$$u_r = u_{r,p} + u_{r,h}$$

$$u_r = -(1-\nu^2)\frac{\rho\omega^2 r^3}{8E} + c_1 r + \frac{c_2}{r}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Disks of constant thickness

From the calculated displacement,

$$u_r = -(1-v^2)\frac{\rho\omega^2 r^3}{8E} + c_1 r + \frac{c_2}{r}$$

from stress-displacement relations,

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\frac{du_r}{dr} + v\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\frac{u_r}{r} + v\frac{du_r}{dr}\right)$$

$$\sigma_{rr} = \frac{E}{1-v^2}\left[\frac{-(3+v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 - (1-v)\frac{c_2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left[\frac{-(1+3v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 + (1-v)\frac{c_2}{r^2}\right]$$

Mechanics of Solids: Yielding in rotating disks

Rotating Annular Disks of constant thickness

Stresses due to rotation without pressure

$$\sigma_{rr}\big|_{r=r_i} = 0; \qquad \sigma_{rr}\big|_{r=r_e} = 0$$

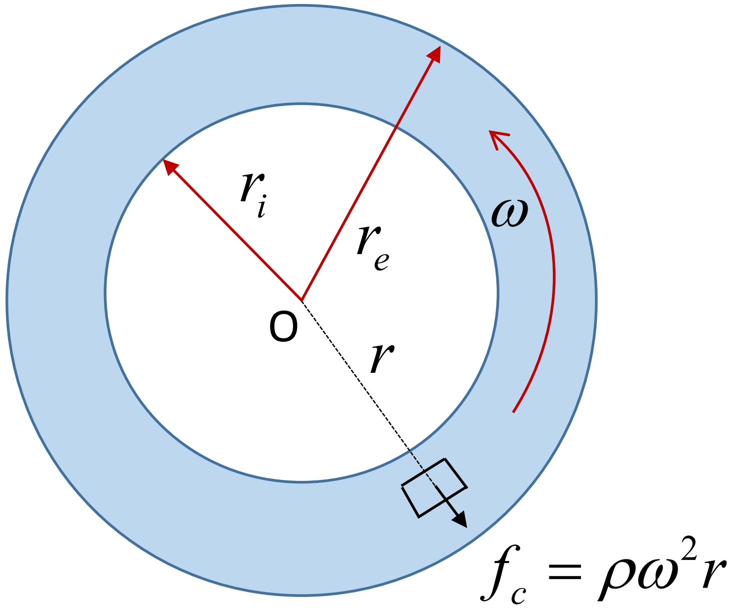
$$\sigma_{rr} = \frac{E}{1-v^2}\left[\frac{-(3+v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 - (1-v)\frac{c_2}{r^2}\right]$$

$$0 = -\rho\omega^2 \frac{r_i^2}{E}\frac{(1-v^2)(3+v)}{8} + (1+v)c_1 - (1-v)\frac{c_2}{r_i^2}$$

$$0 = -\rho\omega^2 \frac{r_e^2}{E}\frac{(1-v^2)(3+v)}{8} + (1+v)c_1 - (1-v)\frac{c_2}{r_e^2}$$

$$c_1 = \rho\omega^2 \frac{r_i^2 + r_e^2}{E}\frac{(1-v)(3+v)}{8}$$

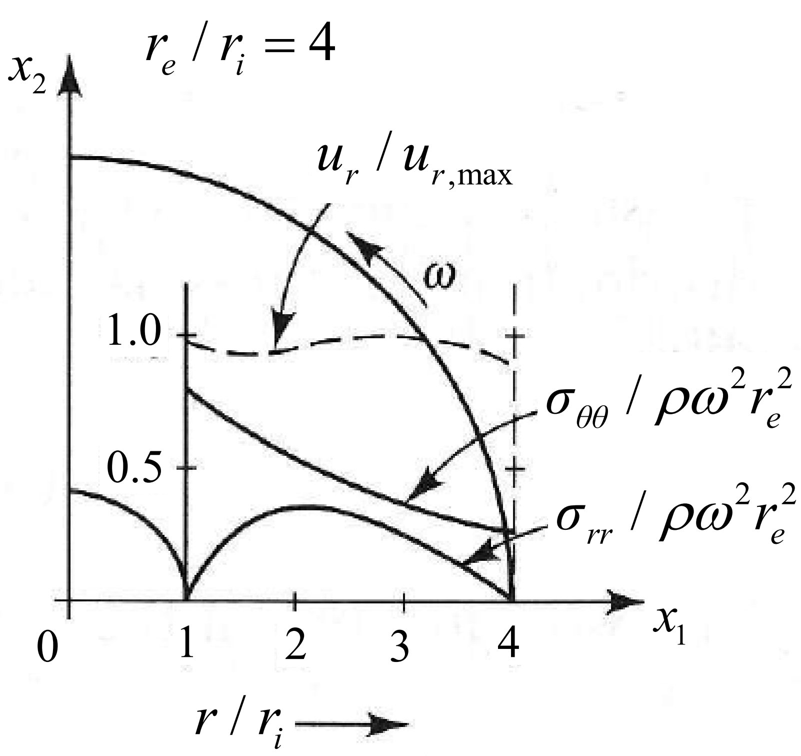
$$c_2 = \rho\omega^2 \frac{r_i^2 r_e^2}{E}\frac{(1+v)(3+v)}{8}$$



Mechanics of Solids: Yielding in rotating disks

Rotating Annular Disks of constant thickness

Stresses due to rotation without pressure,



$$\sigma_{rr} = \frac{(3+v)}{8}\left(r_i^2 + r_e^2 - r^2 - \frac{r_i^2 r_e^2}{r^2}\right)\rho\omega^2$$

$$\sigma_{\theta\theta} = \frac{(3+v)}{8}\left(r_i^2 + r_e^2 - \frac{1+3v}{3+v}r^2 + \frac{r_i^2 r_e^2}{r^2}\right)\rho\omega^2$$

maximum radial stress at $r = \left(r_i r_e\right)^{1/2}$

Maximum elastic stress at $r = r_i$

$$\sigma_{\theta\theta} = \frac{(3+v)}{8}\left(r_e^2 + \frac{1-v}{3+v}r_i^2\right)\rho\omega^2$$

$$\omega_0 = \left(\frac{6}{5r_e^2 + r_i^2}\frac{\sigma_Y}{\rho}\right)^{1/2}$$

The criterion of Tresca gives at **initial yielding**

For simplicity we assume To have Aluminum with

$v = 1/3$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Solid Disks of constant thickness

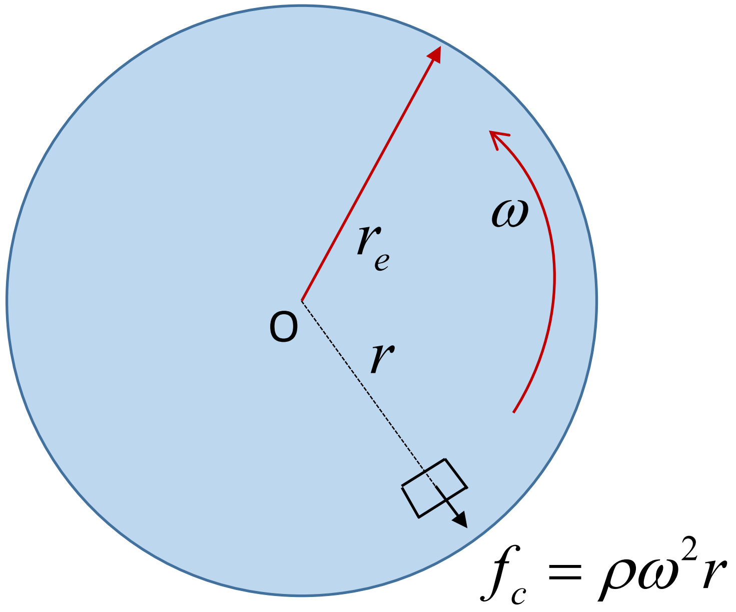
Boundary conditions:

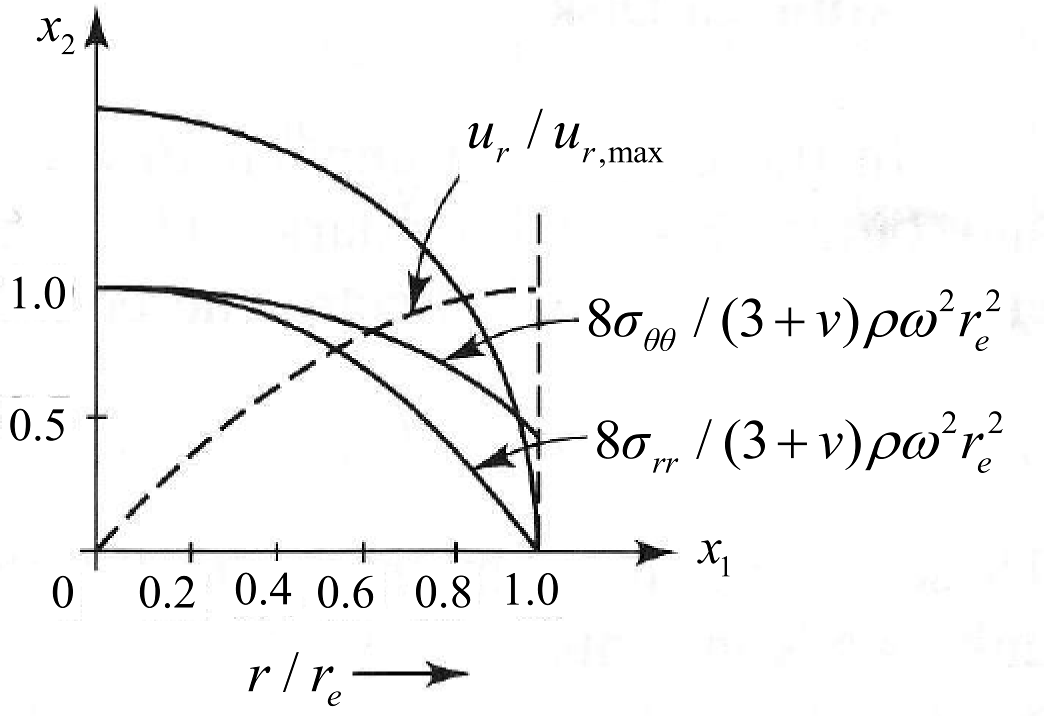
$$r_i = 0; \quad \sigma_{rr}\big|_{r=r_b} = 0; \quad u_r\big|_{r=0} = 0$$

$$c_1 = \rho\omega^2 \frac{r_e^2}{E}\frac{(1-v)(3+v)}{8}; \quad c_2 = 0$$

$$\sigma_{rr} = \frac{3+v}{8}\left(r_e^2 - r^2\right)\rho\omega^2; \quad \sigma_{\theta\theta} = \frac{3+v}{8}\left(r_e^2 - \frac{1+3v}{3+v}r^2\right)\rho\omega^2$$

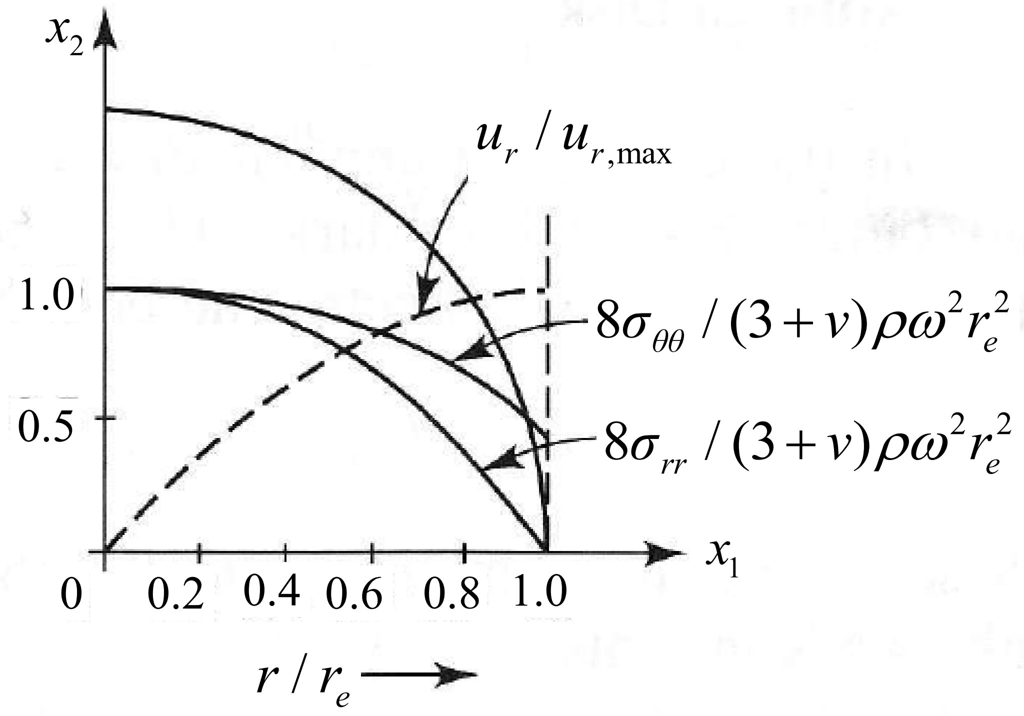
$$u_r = \frac{(1-v)}{8E}\left((3+v)r_e^2 - (1+v)r^2\right)\rho\omega^2 r$$





Mechanics of Solids: Yielding in rotating disks

Rotating Solid Disks of constant thickness



$$\sigma_{rr} = \frac{3+v}{8}\left(r_e^2 - r^2\right)\rho\omega^2$$

$$\sigma_{\theta\theta} = \frac{3+v}{8}\left(r_e^2 - \frac{1+3v}{3+v}r^2\right)\rho\omega^2$$

Maximum stresses are at the center *r* = 0

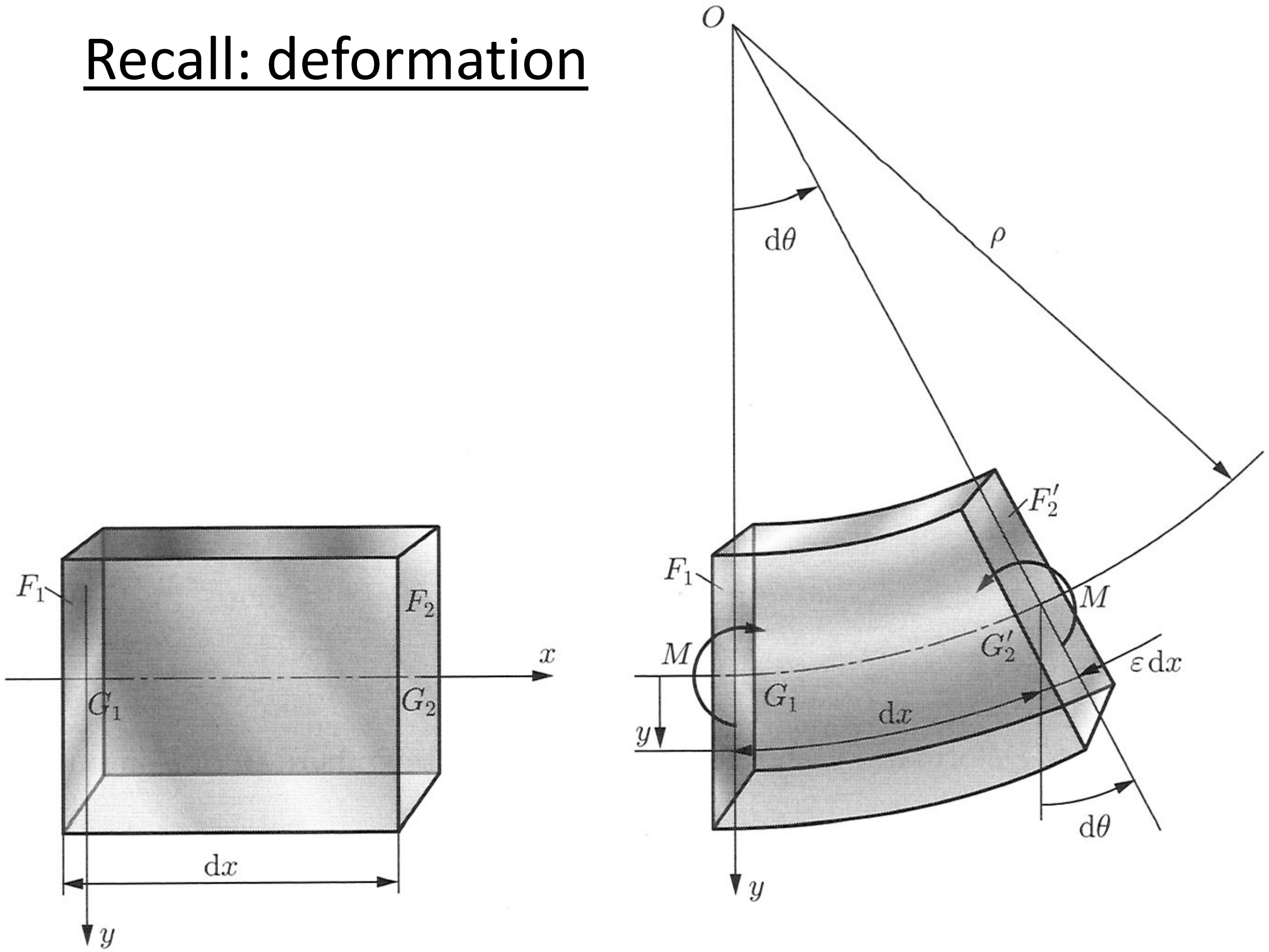
$$\sigma_{\theta\theta} = \sigma_{rr} = \frac{3+v}{8}\left(r_e^2\right)\rho\omega^2$$

Use this equation for design.
Fore example with $\sigma_{\theta\theta} = \sigma_Y$
in Tresca criterion

At first yield $\sigma_Y = \frac{3+v}{8}\left(r_e^2\right)\rho\omega^2$

Pure Bending of straight prismatic beams

<u>Recall: deformation</u>



$$dx = \rho d\theta \Rightarrow \frac{1}{\rho} = \frac{d\theta}{dx}$$

$$\varepsilon dx = y d\theta$$

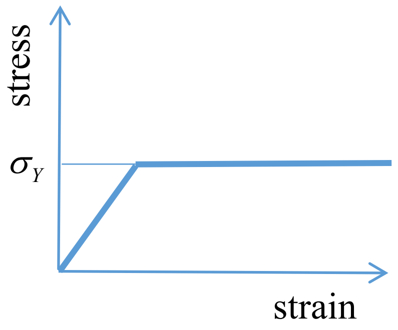
$$\varepsilon = y\frac{d\theta}{dx} = y\frac{1}{\rho}$$

$$\sigma = E\varepsilon = Ey\frac{d\theta}{dx} = E\frac{y}{\rho}$$

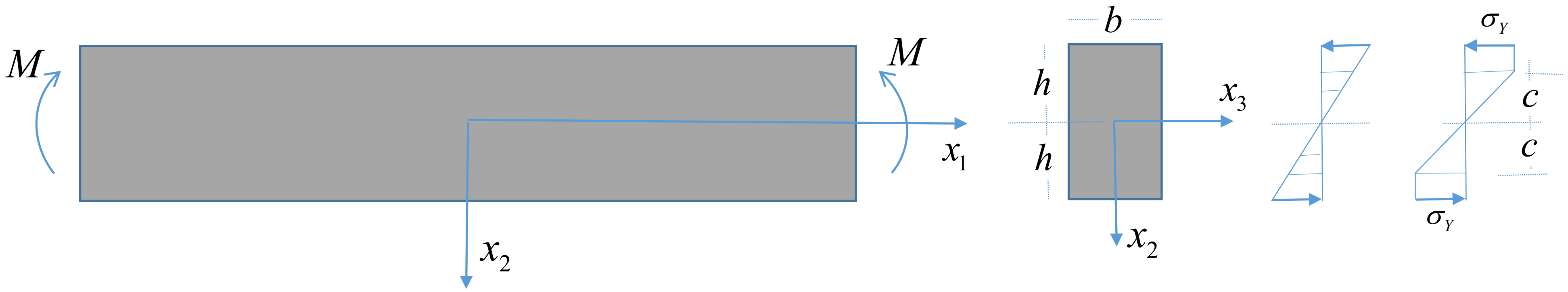
Figure 6.3

Mechanics of Solids: Elastic-perfectly plastic response of a beam

Beam under pure bending
Material is elastic – perfectly plastic
We assume that yielding in tension
and compression are the same



Problem: determine the stress distribution in a beam under pure tension:

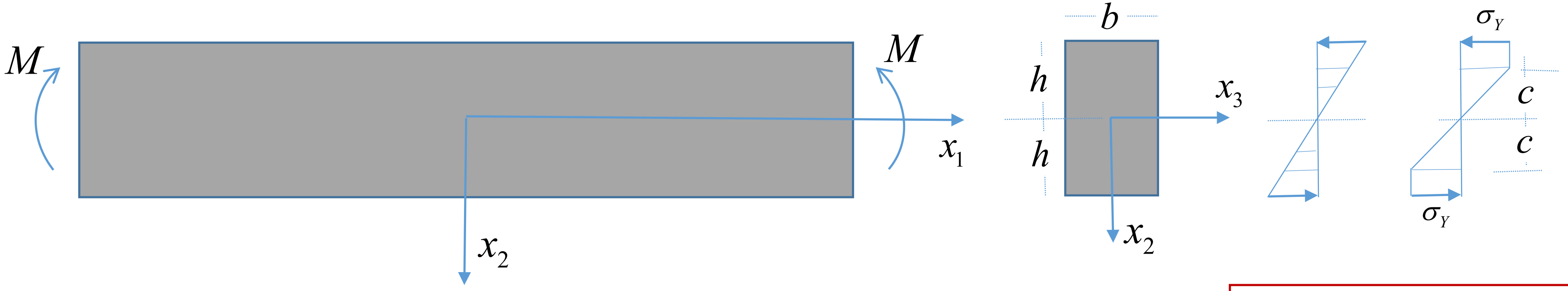


We assume that plasticity has spread up
to a distance from the neutral axis.
Elastic region is

Mechanics of Solids: Elastic-perfectly plastic response of a beam

Beam under pure bending

Material is elastic – perfectly plastic



In the elastic part $\sigma_{11} = E\varepsilon_{11} = E\dfrac{x_2}{\rho}$

At the transition $\sigma_Y = E\varepsilon_{11} = E\dfrac{c}{\rho}$

Equilibrium $M = 2\int_0^c \sigma_{11} x_2 \left(b dx_2\right) + 2\int_c^h \sigma_Y x_2 \left(b dx_2\right)$

$$= 2\int_0^c E\frac{x_2^2}{\rho}\left(b dx_2\right) + 2\int_c^h \sigma_Y x_2 \left(b dx_2\right) = b\sigma_Y\left(h^2 - c^2/3\right)$$

Limiting cases:

At the start of yielding $c = h$

$$M = b\sigma_Y\left(h^2 - c^2/3\right) = \frac{2bh^2}{3}\sigma_Y$$

Full yield $c = 0$

$$M = b\sigma_Y\left(h^2 - c^2/3\right) = bh^2\sigma_Y$$

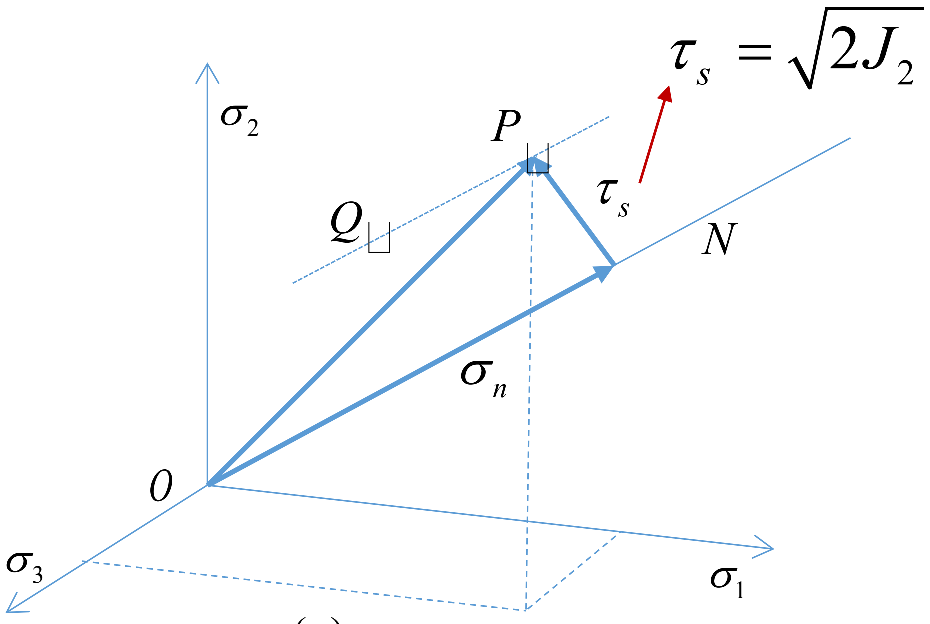
$\tau_s = \sqrt{2J_2}$

Mechanics of Solids: Elastic-plastic response

v Mises criterion: $J_2 = \frac{1}{3}\sigma_Y^2$

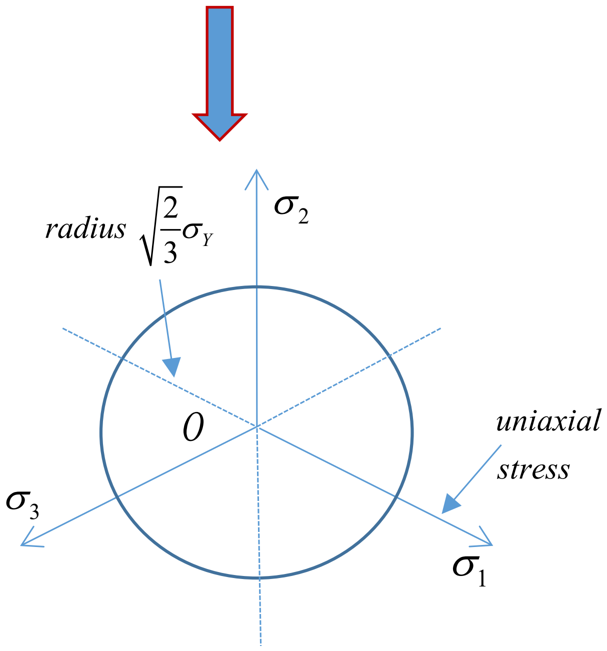
or $\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right]=2\sigma_Y^2$

Octahedral shear stress:



(a)

On the π plane



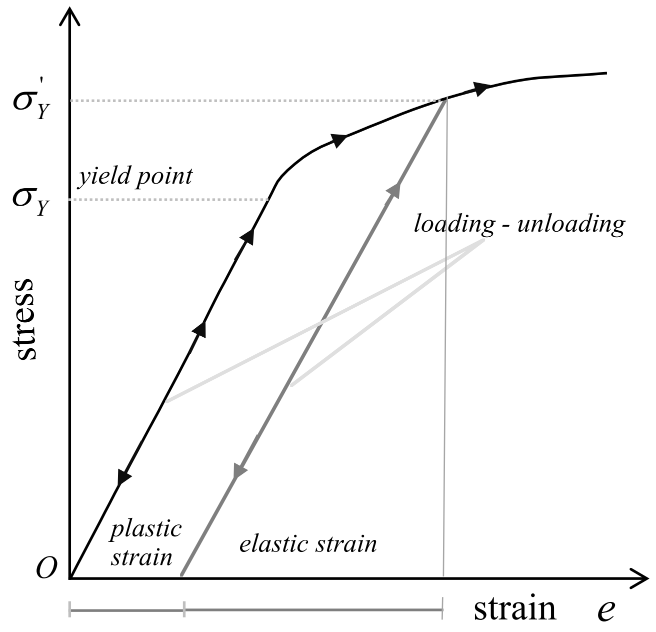
(b)

Mechanics of Solids: Elastic-plastic response

strain-hardening, or ***work hardening***

1. Upon loading, a virgin material will yield when the yield criterion is satisfied.
2. For a perfectly plastic the yield stress and yield surface remain the same upon further loading.
3. In several material, the stress-strain curve rises and thus, the yield stress increases upon further loading.
4. This phenomenon is called *strain-hardening,* or *work hardening*.

➡ As a result the yield surface changes upon loading beyond the yield limit.



Yield function: $f\left(\sigma_{ij}\right)=K$

Loading: $f\left(\sigma_{ij}\right)=K,\quad \dfrac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} > 0$

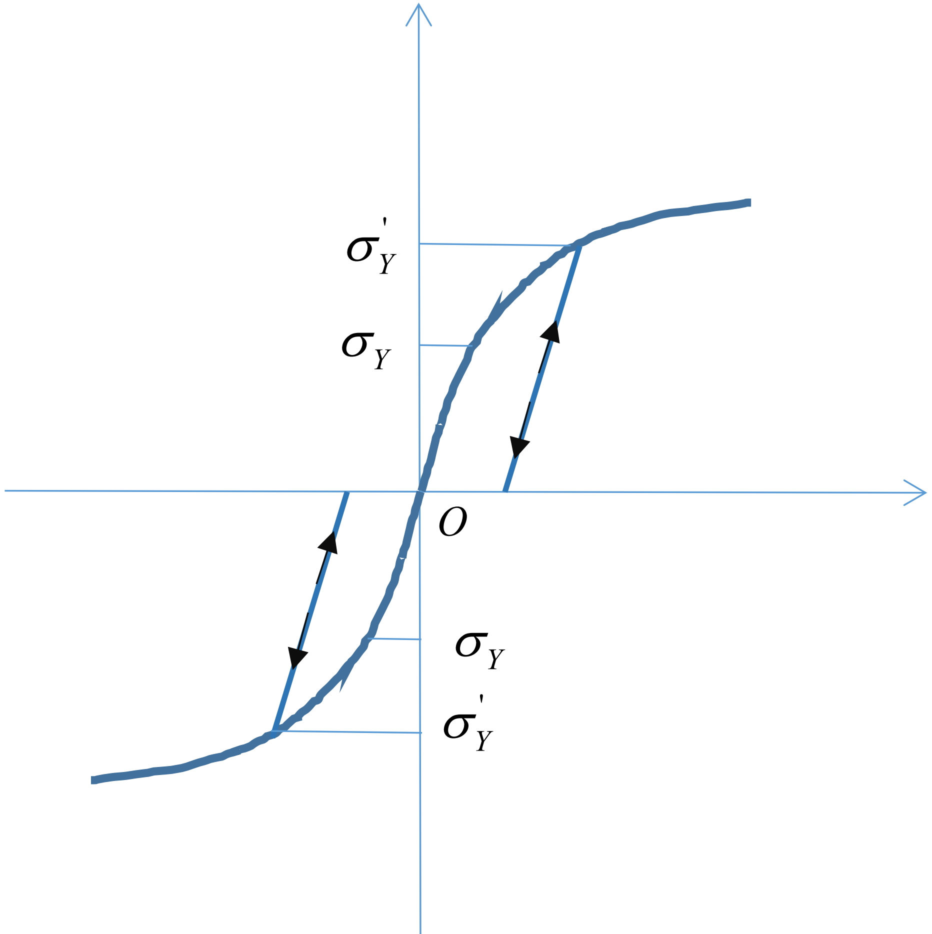
Neutral Loading: $f\left(\sigma_{ij}\right)=K,\quad \dfrac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} = 0$

Unloading: $f\left(\sigma_{ij}\right)=K,\quad \dfrac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} < 0$

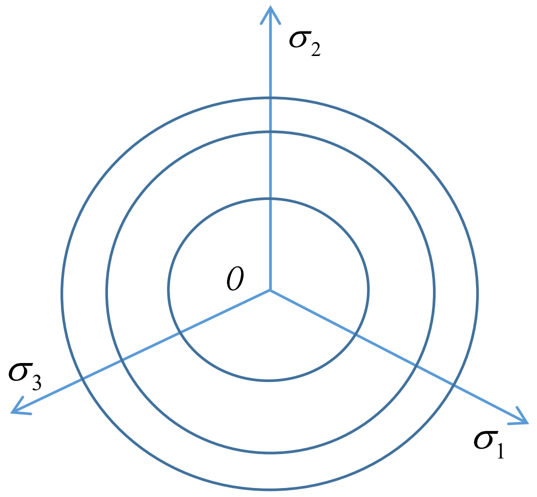
Mechanics of Solids: Elastic-plastic response

Isotropic strain-hardening

The yield stress in tension and compression is assumed the same.



For the v Mises yield criterion, the circle on the π plane expands uniformly and remains concentric.



(*a*)

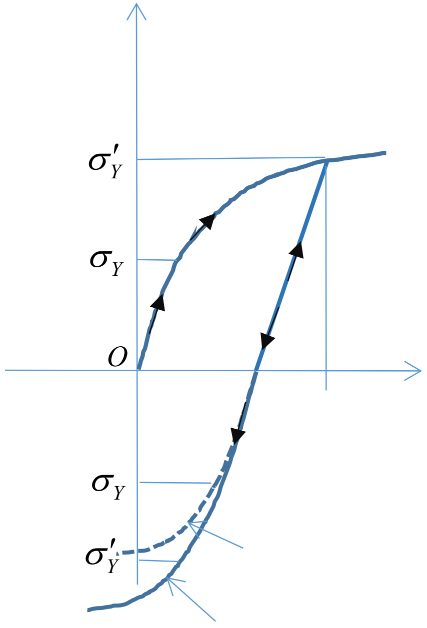
Mechanics of Solids: Elastic-plastic response

Bauschinger Effect, kinematic hardening

Experimental results show that yield is tension is not always the same in compression.

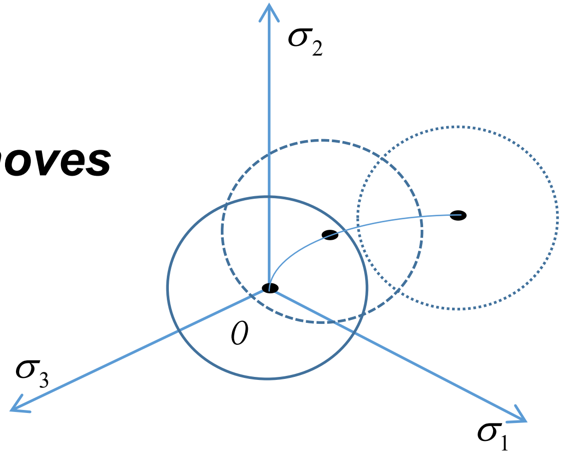
The increase of yield in tension results in a decrease of yield in compression:

This is the so called called ***Bauschinger Effect***



(b)

As a consequence, the yield surface moves in stress space.



(b)

In reality both phenomena (isotropic and kinematic hardening appear simultaneously)

Mechanics of Solids: Elastic-plastic response

Plastic Strains

During loading beyond the yield point, elastic and plastic strains are produced:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$$

The elastic part can be expressed via Hook's law

$$d\varepsilon_{ij}^{e} = S_{ijkl}\, d\sigma_{kl}$$

$$or \qquad d\sigma_{ij} = C_{ijkl}\, d\varepsilon_{kl}^{e}$$

Two theories to model plastic strains.
incremental* or *flow theories of plasticity: They relate plastic strain increments to current stress level. The increments of strains are computed throughout the loading history and expressed in terms of the current stress level. To determine total plastic strains, we integrate the incremental stress-strain relations over the history of loading.
total* or *deformation theories of plasticity: Here the total strain components are related to the current stress.

Plastic Strains

Prandtl-Reuss equations

$$\frac{d\varepsilon_{11}^{p}}{s_{11}}=\frac{d\varepsilon_{22}^{p}}{s_{22}}=\frac{d\varepsilon_{33}^{p}}{s_{33}}=\frac{d\varepsilon_{12}^{p}}{s_{12}}=\frac{d\varepsilon_{23}^{p}}{s_{23}}=\frac{d\varepsilon_{31}^{p}}{s_{31}}=d\lambda\geq 0$$

$$d\varepsilon_{ij}^{p}=s_{ij}d\lambda$$

$$d\varepsilon_{11}^{p}+d\varepsilon_{22}^{p}+d\varepsilon_{33}^{p}=0$$

use $s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}$

To express them in terms of the stress components

$$d\varepsilon_{11}^{p}=d\lambda s_{11}=\frac{2}{3}d\lambda\left[\sigma_{11}-\frac{1}{2}(\sigma_{22}+\sigma_{33})\right]$$

....

$$d\varepsilon_{12}^{p}=d\lambda s_{12}=d\lambda\sigma_{12}$$

....

Plastic Strains
Prandtl-Reuss equations

With the flow rule known, we can have

$$d\varepsilon_{11} = d\varepsilon_{11}^{e} + d\varepsilon_{11}^{p} = \frac{1}{E}\left[d\sigma_{11} - v(d\sigma_{22} + d\sigma_{33})\right] + \frac{2}{3}d\lambda\left[\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})\right]$$

....

$$d\varepsilon_{12} = d\varepsilon_{12}^{e} + d\varepsilon_{12}^{p} = \frac{1+v}{E}d\sigma_{12} + d\lambda\sigma_{12}$$

....

$$d\varepsilon_{ij} = \frac{1+v}{E}d\sigma_{ij} - \frac{v}{E}\delta_{ij}d\sigma_{kk} + d\lambda s_{ij}$$

When the elastic strains are neglected (very small compared to the plastic strains), the remaining relations are called ***Lévy-Mises equations***

Mechanics of Solids: Elastic-plastic response

Plastic Strains
Prandtl-Reuss equations

To identify $d\lambda$ we start with: $\dfrac{d\varepsilon_{11}^p}{s_{11}} = \dfrac{d\varepsilon_{22}^p}{s_{22}} = \dfrac{d\varepsilon_{33}^p}{s_{33}} = \dfrac{d\varepsilon_{12}^p}{s_{12}} = \dfrac{d\varepsilon_{23}^p}{s_{23}} = \dfrac{d\varepsilon_{31}^p}{s_{31}} = d\lambda \geq 0$

We replace: $s_{ij} = \sigma_{ij} - \dfrac{1}{3}\sigma_{kk}$

Define: $\sigma_e = \dfrac{1}{\sqrt{2}}\left[\left(\sigma_{11}-\sigma_{22}\right)^2 + \left(\sigma_{22}-\sigma_{33}\right)^2 + \left(\sigma_{33}-\sigma_{11}\right)^2 + 6\left(\sigma_{12}\right)^2 + 6\left(\sigma_{23}\right)^2 + 6\left(\sigma_{31}\right)^2\right]^{1/2}$

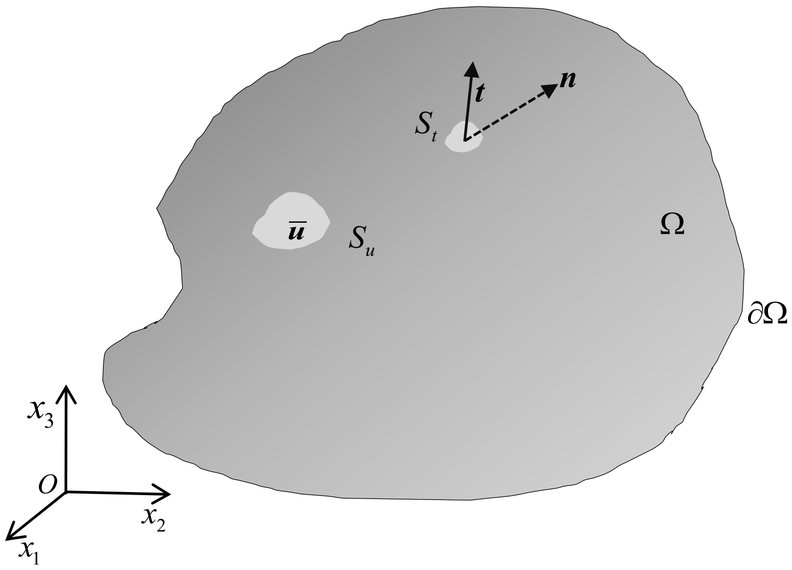
$$d\varepsilon_p = \frac{\sqrt{2}}{3}\left[\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2 + 6\left(d\varepsilon_{12}^p\right)^2 + 6\left(d\varepsilon_{23}^p\right)^2 + 6\left(d\varepsilon_{31}^p\right)^2\right]^{1/2}$$

$$\Rightarrow \boxed{d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e}s_{ij}}$$

For the details see Appendix C: Notes on Plasticity Theory

Mechanics of Solids: Elastic-plastic response

ELASTOPLASTIC STRESS ANALYSIS



When elastic and plastic strains of the same order exist in a body, we talk about elastoplastic problems. Three elements to consider:

1. **Elastic region**
2. Plastic region
3. Elastic-plastic interface

1. Elastic region

Equilibrium: → $\sigma_{ij,j}(x_k) + f_i = 0$

Strain-displacement relations: → $\varepsilon_{ij}(x_k) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Constitutive Equations : → $\varepsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\delta_{ij}\sigma_{kk}$

Compatibility equations:

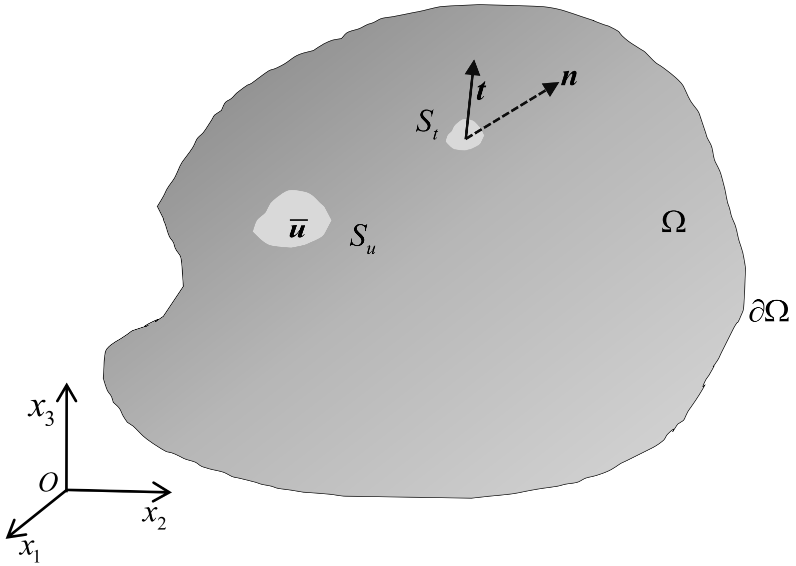
Boundary conditions:

on S_t → $\sigma_{ij}(x_k)n_i(x_k) = t_i(x_k)$

on S_u → $u_i(x_k) = \overline{u}_i(x_k)$

Mechanics of Solids: Elastic-plastic response

ELASTOPLASTIC STRESS ANALYSIS



When elastic and plastic strains of the same order exist in a body, we talk about elastoplastic problems. Three elements to consider:

1. Elastic region
2. **Plastic region**
3. **Elastic-plastic interface**

2. Plastic region

Equilibrium: → $\sigma_{ij,j}(x_k) + f_i = 0$

Stress-strain increment relations

→ $d\varepsilon_{ij}^p = s_{ij} d\lambda$ or $d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e} s_{ij}$

Yield condition: V. Mises

$$\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)\right] = 2\sigma_Y^2$$

Boundary conditions on plastic domain (when it exists)

3. Elastic-plastic interface

Continuity of stresses and displacements